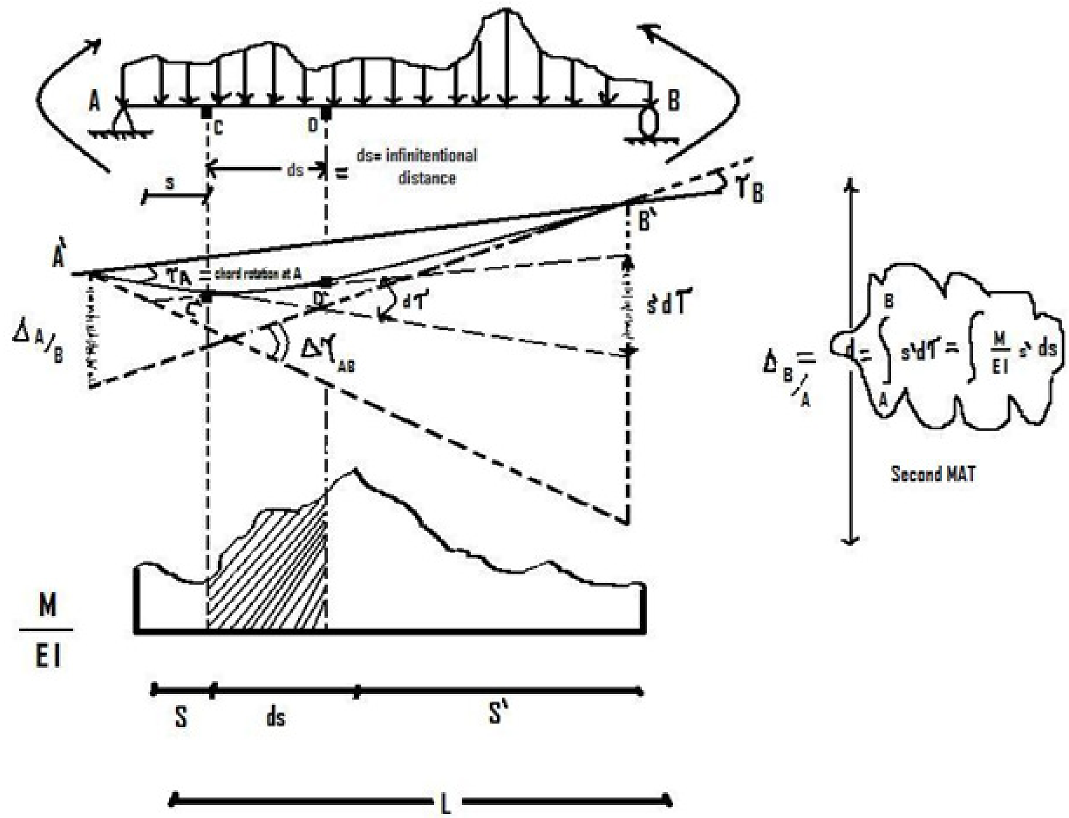
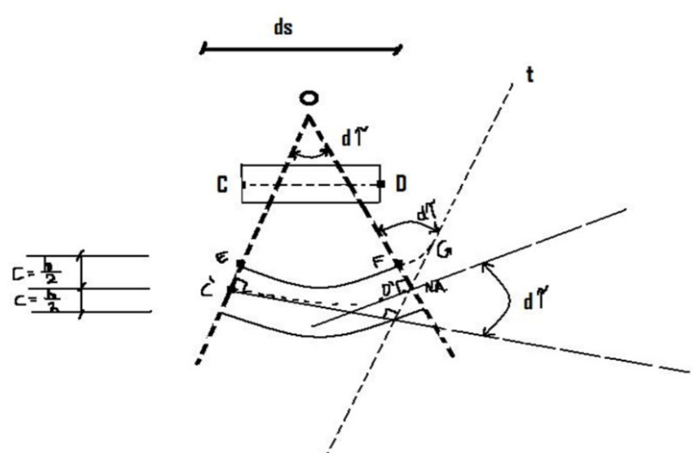
Duc T. Nguyen

Deflection Computation: Moment Area Theorems (M.A.T.)





- From point D', draw line D't // OC'
- Consider deformed beam segment C'D' on the neutral Axis (N.A.). The angle between 2 tangent lines at C' and D' is d τ

Hence: $\widehat{C'OD'} = \widehat{OD't} = d\tau$

- Also curve line EFG is parallel to curve line C'D'

Hence: EFG $\equiv$ ds

- FG = deformation = εds = $(\frac{\sigma_c}{E})$ ds

Also, FG = Cd τ

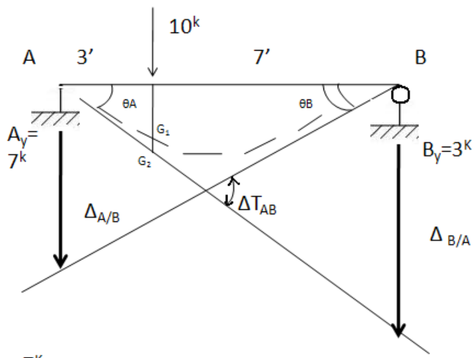
- Hence $\left(\frac{\sigma_c}{E}\right) \mathrm{ds} = \mathrm{C} d\tau$
  - $\left(\frac{\mathrm{MC}}{\mathrm{IE}}\right) \mathrm{ds} = \mathrm{C}\, d\tau$

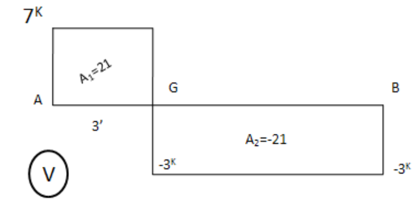
  or $d\tau = \frac{\mathrm{M}}{\mathrm{EI}} \mathrm{ds}$ => $\Delta\tau_{AB} = \int \frac{\mathrm{Mds}}{\mathrm{EI}}$ => 1^{st} M.A.T.

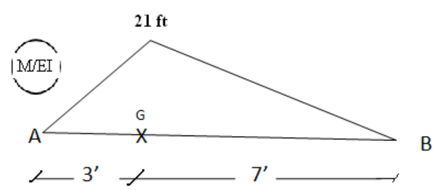
Notes:

The first and the second Moment Area Theorems (M.A.T.) will **NOT** directly give the rotation and displacement of a joint on the beam (or frame).
However, M.A.T will compute the **change in angle** (such as $\Delta\tau_{AB}$) between any 2 points (such as A', B') on the beam, and compute the **vertical** distance (such as d or $\Delta_{B/A}$) between 2 tangent lines (at A', B') on the beam.

Example 1: Use Moment Area Theorems (MAT), find the vertical displacement at point G (= δ_G?) and the rotations θ_a and θ_b at points A and B.







$\delta_G \equiv GG_1 = (GG_2) - G_1G_2 = (3/10 * \Delta_{B/A}) - \Delta_{G/A}$

Where:

$\Delta_{B/A} \overset{2nd\ MAT}{=} (\text{Area under } \frac{M}{EI} \text{diagram})_{Between\ B\ and\ A} *$ (S' = distance from B to centroid of area)

Or: $\Delta_{B/A} = (\frac{1}{2} * 21^{Kft} * 7') * (S'_2 = \frac{2}{3} * 7') + (\frac{1}{2} * 21^{Kft} * 3') * (S'_1 = 7' + \frac{1}{3} * 3')$

$$\Delta_{B/A} = \frac{595}{EI}$$

And $\Delta_{G/A} = (\text{area under M/EI diagram})_{between\ B\ and\ A} *$ (S' = distance from to centroid of area)

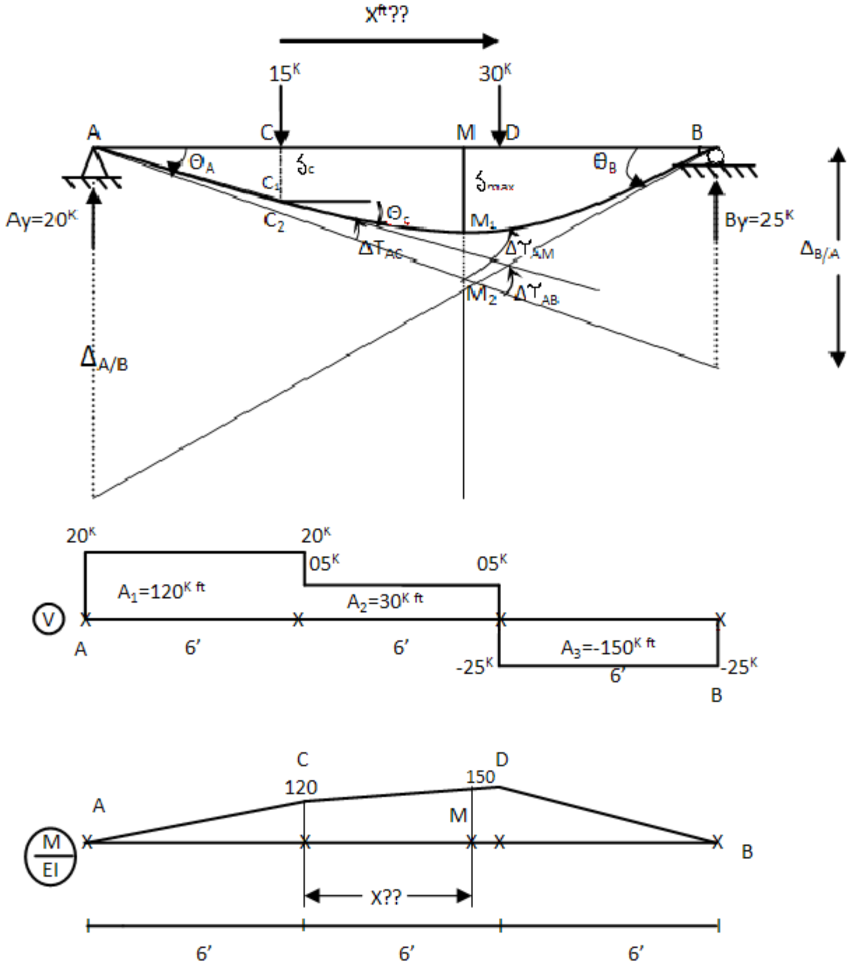
Or $\Delta_{G/A} = (\frac{1}{2} * 21^{Kft} * 3') * (S' = \frac{1}{3} * 3') = \frac{31.5}{EI}$

Thus: $\delta_G \equiv GG_1 = (\frac{3}{10} * \frac{595}{EI}) - \frac{31.5}{EI} = \frac{147}{EI} = \delta_G$

$\theta_A = \text{small angle} = \frac{\Delta_{B/A}}{10'} = \frac{59.5}{EI} = \theta_A$

$\theta_B = \text{small angle} = \frac{\Delta_{A/B}}{10} = (\frac{1}{10})\{\left(\frac{1}{2} * 3' * 21^{Kft}\right) * \left(\frac{2}{3} * 3'\right) + \left(\frac{1}{2} * 7' * 21^{Kft}\right)\left(3' + \frac{1}{3} * 7'\right)\}$

Example 2: Use M.A.T to find $\theta_A, \theta_B, \theta_C, \delta_C, \delta_{max}$



$$\theta_A \cong \frac{\Delta_{A/B}}{18\text{ ft}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where:

$\Delta_{B/A}$ = (Area) * (S' = distance **from B** to centroid area)

$$\Delta_{B/A} = \begin{cases} (\frac{1}{2} 6' * 120) * (S_1' = 12^{ft} + \frac{1}{3} * 6ft) \\ + (\frac{1}{2} * 6' * 30) * (S_2' = 6^{ft} + \frac{1}{3} * 6ft) \\ + (6' * 120) * (S_3' = 6ft + \frac{1}{2} * 6ft) \\ + (\frac{1}{2} * 6' * 150) * (S_4' = \frac{2}{3} * 6ft) \end{cases}$$

or $\Delta_{B/A} = \frac{+14,040}{EI}$ ----------------(1^B)

So: $\theta_A = \frac{\Delta_{B/A}}{18^{ft}} = \boxed{\frac{780}{EI} = \theta_A}$

Similarly, $\theta_B \cong \frac{\Delta_{A/B}}{18^{ft}}$ ……………….(2)

Where: $\Delta_{\underline{A}/B}$= (Area under $\frac{M}{EI}$ diagram) $_{\text{Between A \& B}}$ * (S' = distance **from A** to centroid area)

$$\Delta_{A/B} = \begin{cases} (\frac{1}{2} * 6' * 120)\ (S_1' = \frac{2}{3} * 6') + (\frac{1}{2} * 6' * 30)\ (S_2' = 6ft + \frac{2}{3} * 6ft) \\ + (6' * 120)\ (S_3' = 6' + \frac{1}{2} * 6') + (\frac{1}{2} * 6' * 150)\ (S_4' = 12ft + \frac{1}{3} * 6') \end{cases}$$

$$\Delta_{A/B} = (360)\ (4) + (90)\ (10) + (720)\ (9) + (450)\ (14) = \frac{+\,15{,}120}{EI}$$

Hence: $\boxed{\theta_B = \frac{\Delta_{A/B}}{18} = \frac{840}{EI}}$

Remarks:

1. The area between A&B (of the $\frac{M}{EI}$ diagram) has been decomposed into three (3) triangulars and one (1) rectangular
2. We can also calculate θ_B, as following:

$$\theta_B = \overset{\circlearrowleft +}{=} \theta_A + \Delta\tau_{AB}$$ (= angle, rotation at A + change in angle between A&B

Where: $\Delta\tau_{AB} \equiv$(Area of $\frac{M}{EI}$ diagram) $_{\text{Between A\&B}}$ = $\begin{cases} (\frac{1}{2} * 6' * 120) + (\frac{1}{2} * 6' * 30) \\ + (6' * 120) + (\frac{1}{2} * 6' * 150) \end{cases}$

Or $\Delta\tau_{AB} = \frac{1620}{EI}$

Hence: $\theta_{\mathbf{B}} = (\theta_{\mathbf{A}} \overset{\nearrow}{=}_{cw} - \frac{780}{EI}) + (+\frac{1620}{EI} = \Delta\tau_{AB}) = \boxed{\frac{+840}{EI} = \theta_{\mathbf{B}}}$

ccw ccw

- Also : $\theta_C \overset{\circlearrowleft +}{=} \theta_A + \Delta\tau_{AC}$ ______________ (3)

Where $\theta_A = \frac{\overset{cw}{-}780}{EI}$

And $\Delta\tau_{AB} = 1^{st}$ M.A.T = (area under $\frac{M}{EI}$ diagram)$_{\text{between A\&C}}$

$$\Delta\tau_{AC} = (\frac{1}{2} * 6' * 120) = +\frac{360}{EI}$$

Because CCW (see picture previous page)

Hence $\theta_C = (\frac{-780}{EI}) + (\frac{+360}{EI}) = \boxed{\frac{\overset{cw}{-}420}{EI} = \theta_c}$

- From the figure, one has:

$\delta_c \equiv CC_1 = (CC_2) - [C_1C_2]$ ______________ (4)

Or $\delta_c = \; = (\frac{6\text{ ft}}{18\text{ft}} * \Delta_{B/A}) - [\Delta_{C/A}]$ ______________ (5)

Where: $\Delta_{C/A} = 2^{nd}$ M.A.T. = (area under $\frac{M}{EI}$)$_{\text{Between C \& A}}$ *
(S' = distance from C to centroid area)

Or $\Delta_{C/A} = (\frac{1}{2} * 6' * 120) * (S' = \frac{1}{3} * 6') = \frac{720}{EI}$

Hence: $\delta_c = (\frac{6}{18} * \frac{14{,}040}{EI}) - \left[\frac{720}{EI}\right] = \boxed{\frac{3960}{EI} = \delta_c}$

- To find δ_{max} (located at point M on the beam)

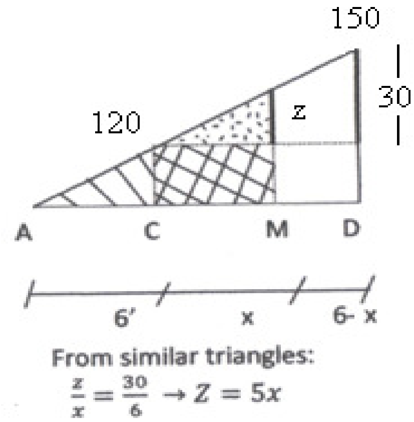
Notes → Point M (with max. deflection) should be close to 30^K load

→ Rotation at point M (θ_M) should be zero

We have: $\theta_M \overset{\circlearrowleft +}{=} \theta_A + \Delta\tau_{AM}$ ________ (6)

$0 = \left(-\frac{720}{EI}\right) + \Delta\tau_{AM}$ ________ (7)

Where:



$$\Delta\tau_{AM} = \left(\text{Area under } \frac{M}{EI}\right)_{\text{Between A\&M}}$$

$$\Delta\tau_{AM} = \begin{cases} \left(\frac{1}{2} * 6' * 120\right) \\ + (1/2 * x * [z = 5x]) \\ +(120 * x) \end{cases}$$

$\Delta\tau_{AM} = (360) + (2.5x^2) + (120x)$ ________ (8)

Substitute Eq (8) into Eq (7) to obtain:

$$0=\left(-\frac{780}{EI}\right) + \left(\frac{+2.5x^2+120x+360}{EI}\right) \boxed{\rightarrow x = 3.28 \text{ ft}}$$

Thus: $\delta_{max} \equiv (MM_2) - M_1M_2 = \left(\frac{9.26\text{ft}}{18\text{ft}} * \Delta_{B/A}\right) - [\Delta_{M/A}]$ ________ (9)

Similar triangles

Where: $\Delta_{\mathbf{M/A}}$ (Use 2nd M.A.T.) = (area under $\frac{M}{EI}$)$_{\text{Between M \& A}}$ *(S' = distance from M to centroid area)

$$\Delta_{M/A} = \begin{cases} \left(\frac{1}{2} * 6' * 120\right) * (S_1' = 3.28' + \frac{1}{3} * 6') \\ + \left(\frac{1}{2} * 3.28 * [z = 5 * 3.28']\right) * (S_2' = \frac{1}{3} * 3.28') \\ + (120 * 3.28) * (S_3' = \frac{1}{2} * 3.28') \end{cases}$$

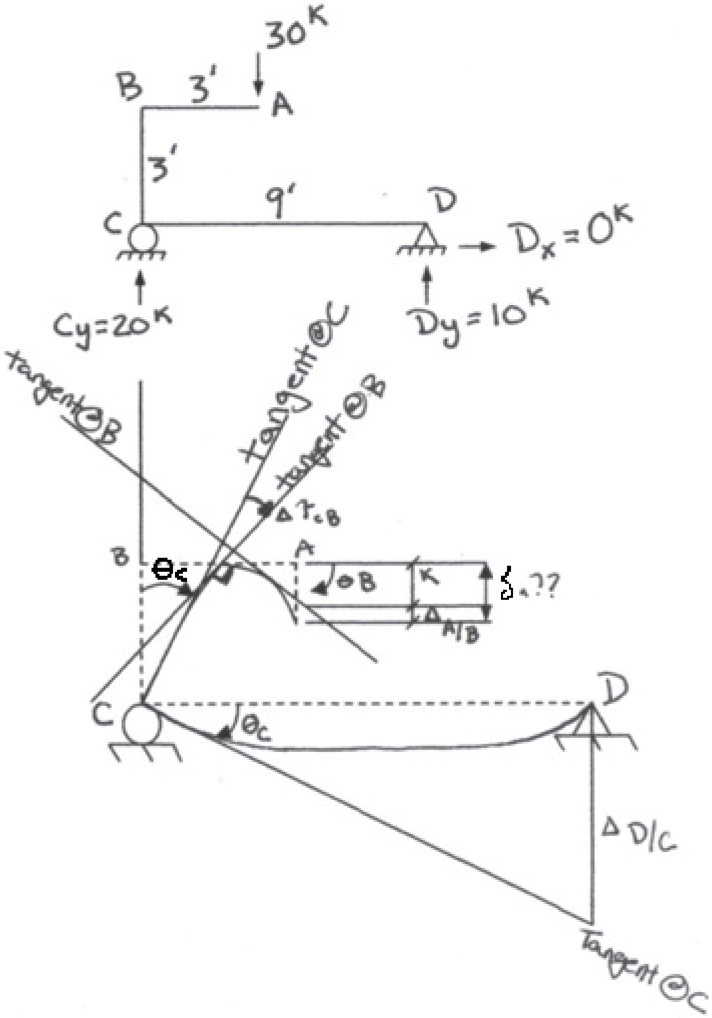
$$\Delta_{M/A} = (360) * (5.28') + (1.64) * (16.4) * (1.09) + (393.6) * (1.64) = \frac{2575.6}{EI} = \frac{2575.6}{EI} \quad (10)$$

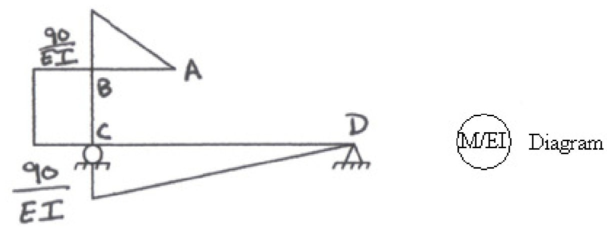
Substituting Eq. (10) of Eq. (1^B) into Eq. (9), to get:

$$\delta_{max} = \frac{9.28^{ft}}{18ft} * \frac{14{,}040}{EI} - [\frac{2575.6}{EI}]$$

$$\boxed{\delta_{max} = \frac{4662.8}{EI}}$$

Example 3 Use Moment Area Theorems (M.A.T.), find the vertical deflection at joint A (δ_A) of the frame structure.





Moments Plotted on Tension Side

Strategies:

- **Step1:** Compute $\Delta_{D/C}$ (use 2nd M.A.T)
- **Step 2:** $\theta_C \cong \frac{\Delta_{D/C}}{9\text{ ft}}$
- **Step 3**: Compute $\Delta\tau_{CB}$ = Change in angle between tangent lines @ C & B (used 1st MAT)
- **Step 4:** $\theta_B = \theta_C + \Delta\tau_{C/B}$
- **Step 5:** $K \cong (3^{ft}) * \theta_B$
- **Step 6:** Compute $\Delta_{A/B}$ (use 2nd MAT)
- **Step 7:** $\delta_A = K + \Delta_{A/B}$

Step 1: $\Delta_{D/C}$ = (area of $\frac{M}{EI})_{D\to C}$) * (S' = distance from D to centroid area)

$= (\frac{1}{2} * \frac{90}{EI} * 9^{ft}) * (S' = \frac{2}{3} * 9^{ft}) = \frac{2430}{EI}$

Step 2: $\theta_C = \frac{2430}{9EI} = \frac{270}{EI}$

Step 3: $\Delta\tau_{CB}$ = (area of $\frac{M}{EI})_{C\to B} = \frac{90}{EI} * 3^{ft} = \frac{270}{EI}$

Step 4: $\theta_B \overset{\circlearrowright +}{=} (\frac{270}{EI}) + (\frac{+270}{EI}) = (\frac{540}{EI})$

Step 5: $K = 3 * (\frac{540}{EI}) = \frac{1620}{EI}$

Step 6: $\Delta_{A/B}$ = (area of $\frac{M}{EI})_{A\to B}$ * (S' = distance from A to centroid area)

$= (\frac{1}{2} * \frac{90}{EI} * 3^{ft}) * (S' = \frac{2}{3} * 3^{ft}) = \frac{270}{EI}$

Step 7: $\delta_A = \frac{1620}{EI} + \frac{270}{EI} = \boxed{\frac{1890}{EI} = \delta_A}$